SO, $\sum_{i=1}^{\infty} \frac{1}{i^2} = \underbrace{\frac{1}{1^2} + \underbrace{\sum_{i=2}^{\infty} \frac{1}{i^2}}_{\text{CONV, IS A NUMBER}}}_{\text{IS A NUMBER}}$ CONV (INTEGRAL TEST)

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~~CONSIDER~~

INTEGRAL TEST

IF $f(x)$ IS POSITIVE, DECREASING AND CONTINUOUS ON $[1, \infty)$

AND $a_n = f(n)$

THEN $\int_1^{\infty} f(x)\,dx$ CONV IF AND ONLY IF $\sum_{i=1}^{\infty} a_i$ CONV

(ALSO MEANS THAT IF $\int_1^{\infty} f(x)\,dx$ DIV,

THEN $\sum_{i=1}^{\infty} a_i$ DIV)

eg.

DOES $\sum_{n=1}^{\infty} n e^{-n}$ CONV/DIV ?

OR $\sum_{i=1}^{\infty} i e^{-i}$

$xe^{-x}$ $\cancel{ne^{-n}} > 0$, $f(x) = xe^{-x}$ IS CONT ON $[1, \infty)$

$f'(x) = e^{-x} - xe^{-x} = \underbrace{(1-x)}_{<0 \text{ ON } (1,\infty)}\underbrace{e^{-x}}_{>0} < 0$ ON $(1, \infty)$

SO $f$ IS DECR ON $[1, \infty)$

CAN USE INTEGRAL TEST

$$\boxed{\int_1^\infty xe^{-x}\,dx = \lim_{N\to\infty}\int_1^N xe^{-x}\,dx}$$

$$= \lim_{N\to\infty} -(x+1)e^{-x}\Big|_1^N$$

$$= \lim_{N\to\infty}\left(-(N+1)e^{-N} - -2e^{-1}\right)$$

OR

$$= \lim_{N\to\infty}\left(2e^{-1} - (N+1)e^{-N}\right) \quad \text{CONV}$$

CONV CONV

SO $\int_1^\infty xe^{-x}\,dx$ CONV

SO $\sum_{i=1}^\infty ie^{-i}$ CONV (INT TEST)

($\int$ TEST)

$$\int u\,dv = uv - \int v\,du$$

| | $u$ | | $dv$ | |
|---|---|---|---|---|
| $\frac{d}{dx}$ $\downarrow$ | $x$ | $+$ | $e^{-x}$ | $\downarrow \int dx$ |
| | $1$ | $-$ | $-e^{-x}$ | |
| | $0$ | | $e^{-x}$ | |

$$-xe^{-x} - e^{-x}$$

$$-(x+1)e^{-x}$$

$$\lim_{N\to\infty}(N+1)e^{-N} \qquad \infty\cdot 0 \text{ INDET}$$

$$\overset{L'H}{=} \lim_{N\to\infty}\frac{N+1}{e^N} \qquad \frac{\infty}{\infty} \text{ INDET L'H}$$

$$\overset{L'H}{=} \lim_{N\to\infty}\frac{1}{e^N}$$

$$= \lim_{N\to\infty} e^{-N} = 0$$

FOR WHAT VALUES OF $p$ DOES $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONV / DIV ?

IF $p=1$, $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$ DIV

IF $p=2$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ CONV

$f(x) = \frac{1}{x^p} > 0$, CONT, DECR IF $p>0$ ON $[1,\infty)$

$$f'(x) = (x^{-p})' = \underbrace{-p}_{\substack{<0 \\ \text{IF } p>0}}\ \underbrace{x^{-p-1}}_{\substack{>0 \\ \text{SINCE } x>0}} < 0 \text{ ON } [1,\infty)$$

CAN USE $\int$ TEST (IF $p>0$)

$$\int_1^{\infty} x^{-p}\,dx = \lim_{N\to\infty} \int_1^N x^{-p}\,dx$$

$$= \lim_{N\to\infty} \frac{1}{-p+1} x^{-p+1} \Big|_1^N$$

$$= \lim_{N\to\infty} \frac{1}{-p+1}\left(N^{-p+1} - 1\right) \quad \text{IF } p \neq 1$$

CONV IF $p>1$

DIV IF $p<1$

$\lim_{N\to\infty} N^{-p+1} = \infty$ (DIV) IF $-p+1>0$

$\lim_{N\to\infty} N^{-p+1}$ IF $-p+1<0$

$= \lim_{N\to\infty} \frac{1}{N^{p-1}}$ $\quad \frac{1}{\infty}$

$= 0$ (CONV)

$-p+1>0$ $\quad -p+1<0$ $\quad$ NOTE: $-p+1=0$

$-p>-1$ $\quad -p<-1$ $\quad p=1$

$p<1$ $\quad p>1$

SO, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ CONV IF } p>1 \qquad \text{DIV IF } p \leq 1 \qquad p \in \mathbb{R}}$ p-SERIES TEST

IF $p=0$: $\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1 = 1+1+1+1+\dots$ DIV BY DIV TEST

$\{1\} \longrightarrow 1$

IF $p<0$

eg. $p=-3$ $\sum_{n=1}^{\infty} \frac{1}{n^{-3}} = \sum_{n=1}^{\infty} n^3$ DIV BY DIV TEST

$\{n^3\} \longrightarrow \infty$

eg. $\sum_{n=1}^{\infty} \frac{7}{\sqrt[3]{n}} \boxed{= 7\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}}$ $\boxed{\text{DIV}}$ BY $\boxed{\text{p-SERIES TEST}}$

$\boxed{p = \frac{1}{3}}\boxed{\leq 1}$

DOES $\sum_{n=1}^{\infty} \frac{n}{2n-1}$ CONV / DIV ?

$f(x) = \frac{x}{2x-1} > 0$, CONT, DECR ON $[1, \infty)$

$f'(x) = \frac{1(2x-1) - x(2)}{(2x-1)^2} = \frac{-1}{(2x-1)^2} < 0$

CAN USE $\int$ TEST

$\int_1^{\infty} \frac{x}{2x-1}\,dx = \ldots$

BUT

$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2-0} = \frac{1}{2} \neq 0$

$\sum_{n=1}^{\infty} \frac{n}{2n-1}$ DIV BY DIV TEST